Heat and Mass transfer

Conduction in a Semi- Infinite Solid

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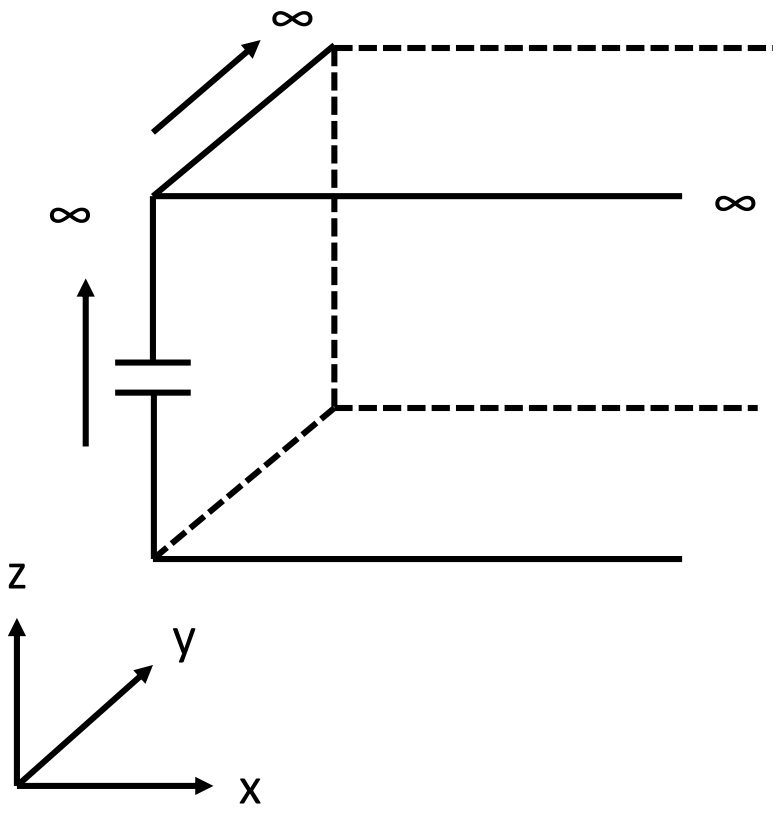


Consider a semi-infinite solid. It is initially at Temperature T_0.

Suddenly at t = 0, the face x = 0 is raised to a temperature T_∞

We require to find the penetration depth of the high temperature.

The governing equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

The initial condition:
$T(x, t) = T_0$ for t = 0

Boundary condition:
$T(0, t) = T_\infty$, for t > 0
$T(\infty, t) = T_0$, for t > 0

Technique:

Similarity transformation of variables

Put $\theta = T - T_\infty$

$\theta = f(x, t, \alpha)$

or, $\theta = f(\eta)$

where $\eta = \dfrac{x}{\sqrt{4\alpha t}}$

now $$\alpha \frac{\partial^2 \theta}{\partial x^2} = \alpha \frac{1}{2\sqrt{\alpha t}} \frac{\partial}{\partial \eta} \left[\frac{1}{2\sqrt{\alpha t}} \frac{\partial}{\partial \eta} \theta \right]$$

or $$\alpha \frac{\partial^2 \theta}{\partial x^2} = \frac{\alpha}{4\alpha t} \frac{\partial^2 \theta}{\partial \eta^2}$$

and $$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta} \left(-\frac{1}{2} \right) \frac{x}{2\sqrt{\alpha t} \quad t}$$

GE $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{1}{4t} \frac{\partial^2 \theta}{\partial \eta^2} + \frac{x}{4\sqrt{\alpha t}\ t} \frac{\partial \theta}{\partial \eta} = 0$$

$$\frac{d^2 \theta}{d\eta^2} + 2\eta \frac{d\theta}{d\eta} = 0$$

let $$\phi = \frac{d\theta}{d\eta}$$

$$\frac{d\phi}{d\eta} + 2\eta\phi = 0$$

$$\frac{d\phi}{\phi} = -2\eta \ d\eta$$

$$\ln \phi = -2\,\eta^2/2 + C_1$$

$$\ln\ \phi = -\eta^2 + \ln\ \mathrm{A}$$

$$\ln\left(\frac{\phi}{A}\right) = -\eta^2$$

$$\phi = Ae^{-\eta^2}$$

$$\frac{d\theta}{d\eta} = Ae^{-\eta^2}$$

$$\int_{\theta(\eta=0)}^{\theta(\eta)} d\theta = \int_{\eta=0}^{\eta} Ae^{-\eta^2} d\eta$$

$$\theta\Big|_{at\ \eta} = \int_{0}^{\eta} Ae^{-\eta^2} d\eta\ + \theta\Big|_{at\ \ \eta=0}$$

$$\theta = A \int_0^{\eta} e^{-\eta^2} d\eta$$

at x -> ∞ $\theta = \theta_0$

or, at η -> ∞ $\theta = T_0 - T_\infty$

Applying the boundary condition, we get

$$T_0 - T_\infty = A \int_0^{\infty} e^{-\eta^2} d\eta = A \frac{\sqrt{\pi}}{2}$$

$$A = \frac{2(T_0 - T_\infty)}{\sqrt{\pi}}$$

$$\theta = \frac{2}{\sqrt{\pi}} (T_0 - T_\infty) \int_0^{\eta} e^{-\eta^2} d\eta$$

$$\left[\begin{aligned} & \int_0^{\infty} e^{-\eta^2} d\eta = \int_0^{\infty} e^{-z} \frac{dz}{2 z^{1/2}} \\ & = \frac{1}{2} \int_0^{\infty} e^{-z} z^{(1/2-1)} dz = \frac{1}{2} \Gamma(1/2) \\ & = \frac{\sqrt{\pi}}{2} \end{aligned} \right.$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta = \text{erf}\ (\eta)$$

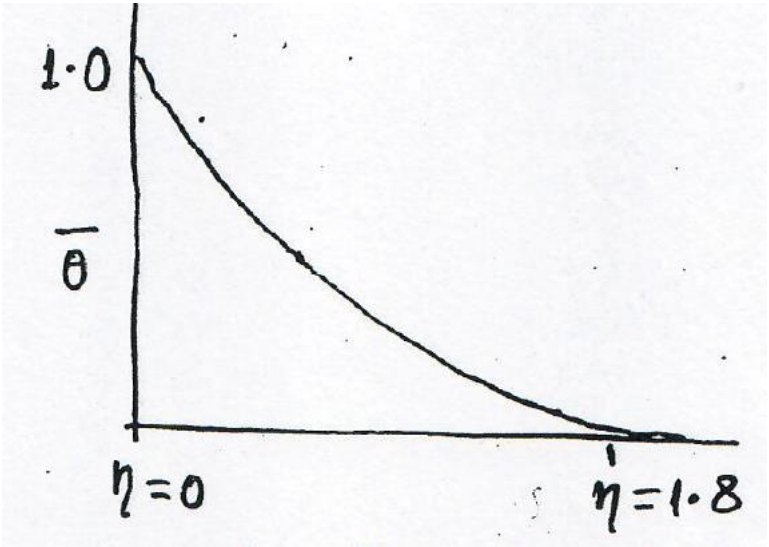
$$\frac{T - T_\infty}{T_0 - T_\infty} = erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$\frac{x}{\sqrt{4\alpha t}}$	$erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$
0.00	0.00
0.02	0.02256
0.04	0.04511
0.06	0.06762
0.10	0.11246
0.12	0.13476
0.14	0.15695
0.16	0.17901
1.8	0.98909

We define a parameter

$$\bar{\theta} = 1 - \frac{T - T_\infty}{T_0 - T_\infty} = \frac{T_0 - T}{T_0 - T_\infty} = \frac{T - T_0}{T_\infty - T_0}$$

Again, $\bar{\theta} = 1 - erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$



$\bar{\theta} = 1$ At η = 0

$\bar{\theta} = 0.01$ At η = 1.8

Thus, at η = 1.8, the body temperature is at initial temperature

Remarks:

1) At $x = 1.8\sqrt{4\alpha t}$ $\bar{\theta} \approx 0$ $T = T_0$

We can say that the temperature perturbation at the surface has penetrated to $x = 1.8\sqrt{4\alpha t}$ and this depth is called the penetration depth.

2) Penetration depth increases as the square root of time “t”.

3) Solid with high α means: $\alpha = k/(\rho C_p)$ = high conductivity/ low heat capacity

More heat is conducted inwards -> Increased penetration depth. Reverse is true for low thermal diffusivity.

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta \qquad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$\frac{\partial T}{\partial x} = (T_0 - T_\infty) \frac{\partial}{\partial x} \left(\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta \right)$$

Now,

$$f = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta$$

$$\frac{df}{dx} = \frac{df}{d\eta} \frac{d\eta}{dx} = \frac{2}{\sqrt{\pi}} \frac{d}{d\eta} \int_0^\eta e^{-\eta^2} d\eta . \frac{d\eta}{dx} \qquad \eta = \frac{x}{\sqrt{4\alpha t}}$$

$$= \frac{2}{\sqrt{\pi}} \frac{e^{-\eta^2}}{\sqrt{4\alpha t}} = \frac{e^{-\eta^2}}{\sqrt{\pi \alpha t}}$$

$$\frac{\partial T}{\partial x} = \frac{(T_0 - T_\infty)\ \ e^{\frac{-x^2}{4\alpha t}}}{\sqrt{\pi \alpha t}}$$

Now, heat transfer at the surface

$$q = -kA \left.\frac{\partial T}{\partial x}\right|_{x=0}$$

$$q = -kA \frac{T_0 - T_\infty}{\sqrt{\pi \alpha t}}$$

Definition of homogeneous and nonhomogeneous problems:

The problem will be referred to as a homogeneous problem when both the differential equation and the boundary conditions are homogeneous.

The definition of Homogeneous and Nonhomogeneous problems:

The problem is homogeneous, if the differential equation and the boundary conditions are Homogeneous.

Homogeneous

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ In region R, for t > 0

$$k\frac{\partial T}{\partial r_i} + hT = 0$$ On boundary S, for t > 0

T = F (r) In region R, for t = 0

Nonhomogeneous

$$\nabla^2 T + \frac{g(r,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

In region R, for t > 0

$$k\frac{\partial T}{\partial r_i} + h\,T = \mathrm{f(r,t)}$$

On boundary S, for t > 0

T = F (r)

In region R, for t = 0

Here the differential equation and boundary conditions are non-homogeneous. Function g (r, t) and f (r, t) do not include T as product.

The error function with argument (x) is defined as

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

where t is a dummy variable. The error function is an odd function, so that

$$\mathrm{erf}(-x) = -\mathrm{erf}(x)$$

In addition, $\mathrm{erf}\,(0) = 0$ and $\mathrm{erf}\,(\infty) = 1$

The complementary error function with argument (x) is defined as

$$erfc\,(x) = 1 - erf\,(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Thank you